Variational Asymptotic Method for Unit Cell Homogenization

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This article presents an overview of a recently developed micromechanics theory, namely, the variational asymptotic method for unit cell homogenization along with its companion code VAMUCH. It is emerging as a general-purpose micromechanics tool for predicting not only the effective properties of heterogeneous materials but also the local fields within the microstructure. The differences between VAMUCH and other micromechanics approaches are articulated. A simple realistic example is used to demonstrate it application in practical situations.

1 Introduction

Composite materials have proved to perform better than conventional materials. The increased complexity of engineering systems at the microlevel, however, greatly complicates the analysis of the macroscopic behavior, which is indispensable for a rational design of these systems. Direct analysis of such systems, although possible, is computationally intensive and unrealistic. Fortunately, most composite materials exhibit statistical homogeneity [11] so that we can define a representative volume element (RVE), which is entirely typical of the whole mixture on average and contains a sufficient number of inclusions for the apparent overall properties to be effectively independent of the boundary conditions [14]. Although different definitions are given for an RVE in the literature [18], we give a practice-oriented definition for an RVE as any block of material the analyst wants to use for the micromechanical analysis to find the effective properties to replace it with an equivalent homogeneous material. The term unit cell (UC) is also used extensively in heterogeneous materials and defined as the building block of the heterogeneous material. In our work, we define UC as the smallest RVE. In other words, an RVE can contain several UCs. These definitions essentially imply that it is the engineer's judgement to determine what should be contained in an RVE or UC. To be consistent with statistical homogeneity, a well-formulated micromechanics model should not depend on the size of an RVE, which means the effective properties obtained from an RVE containing multiple UCs should be the same as those obtained from a UC.

In the past several decades, numerous micromechanical methods have been proposed (see [11] and references cited therein). These include the earliest rules of mixture approaches based on Voigt and Reuss hypotheses which predict the upper and lower bounds, respectively, for the effective elastic properties [13]. For general heterogeneous materials, the difference between these two bounds may be too large to be of practical use. Researchers have proposed various techniques to either reduce the difference between the upper and lower bounds, or find an approximate value between the upper and lower bounds. Typical approaches are the self-consistent model [15] and its generalizations [9, 4], the variational approach of Hashin and Shtrikman [12], third-order bounds [17], the method of cells (MOC) [1, 2] and its variants [19, 3, 26], recursive cell method [5], mathematical homogenization theories (MHT) [6], finite element approaches using conventional stress analysis of an RVE [20], the recently developed variational asymptotic method for unit cell homogenization (VAMUCH) [27, 28, 21, 22, 23], and many others. An extensive review of this field will easily fill volumes and is beyond the scope of this article. Rather, we will focus on providing an overview of VAMUCH including its theoretical foundation, its unique features comparing to other micromechanics approaches, and a realistic example to illustrate how VAMUCH can be used in practical situations. More details can be found in its pertinent publications.

2 Theoretical Foundation

Although different approaches adopt different assumptions in the literature, there are two essential assumptions associated with micromechanics:

- **Assumption 1** The exact solutions of the field variables have volume averages over the UC. For example, if u_i are the exact displacements within the UC, there exist v_i such that

  $$v_i = \frac{1}{\Omega} \int_{\Omega} u_i \, \mathrm{d}\Omega \equiv \langle u_i \rangle \tag{1}$$

  where Ω denotes the domain occupied by the UC and its volume.
- **Assumption 2** The effective material properties obtained from the micromechanical analysis of the UC are independent of the geometry, the boundary conditions, and loading conditions of the macroscopic structure, which means that effective material properties are assumed to be the intrinsic properties of the material when viewed macroscopically.

Please note that these assumptions are not restrictive. The mathematical meaning of the first assumption is that the exact solutions of the field variables are integrable over the domain of UC, which is true almost all the time.

The second assumption implies that we can neglect the size effects and loading effects of the material properties in the macroscopic analysis, which is an assumption often made in conventional continuum mechanics. Of course, the micromechanical analysis of the UC is only needed and appropriate if $h/l \ll 1$ with h as the characteristic size of the UC and l as the the characteristic length scale of the macroscopic behavior of the heterogeneous material. All the other assumptions such as particular geometry shape and arrangement of the constituents, specific boundary conditions, and prescribed relations between local fields and global fields are convenient but not essential. This fact motivated the authors to derive a micromechanics approach which only invokes these two basic assumptions of micromechanics, resulting in the variational asymptotic method for unit cell homogenization (VAMUCH). VAMUCH is founded on the variational asymptotic method (VAM) [8], which is a very powerful mathematical methodology to simplify the procedure of solving physical problems involving small parameters. In contrast to conventional asymptotic methods, VAM carries out asymptotic analysis of the variational statement, synthesizing both merits of variational methods (*viz.*, systematic, simple, and easy to be implemented numerically) and asymptotic methods (*viz.*, without *ad hoc* assumptions). The application of VAM to homogenize isotropic material with periodic cavities [7] is foundational to the development of VAMUCH.

VAMUCH uses three coordinate systems including two Cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$, and an integer-valued coordinate $\mathbf{n} = (n_1, n_2, n_3)$; see Fig. 1. We use x_i as the global coordinates to describe the macroscopic structure and y_i parallel to x_i as the local coordinates to describe the UC. Here and throughout the paper, Latin indices assume 1, 2, and 3 and repeated indices are summed over their range except where explicitly indicated. We choose the origin of the local coordinates y_i to be the geometric center of the UC. To uniquely locate a UC in the heterogeneous material we also introduce integer coordinates n_i. The integer coordinates are related to the global coordinates in such a way that $n_i = x_i/d_i$ with d_i denoting the edge lengths of the UC (no summation over i). It is emphasized that although only two-dimensional (2D) square array is sketched in Fig. 1, VAMUCH is directly applicable to other 2D UCs, and one-dimensional (1D) and three-dimensional (3D) UCs.

The VAMUCH formulation starts from a variational statement of a continuum mechanics description of heterogeneous materials. The second assumption implies that we could obtain the same effective material properties from an imaginary unbounded and unloaded heterogeneous material with the same microstrucutre as the loaded and bounded one. Hence we could derive the micromechanical analysis from a heterogeneous material which could completely occupy the 3D space $\mathcal{R}$ and is composed of infinitely many UCs. For example, if we are dealing with linear elastic behavior, the total potential energy is equal to the summation of the strain energy stored in all the UCs, which is:

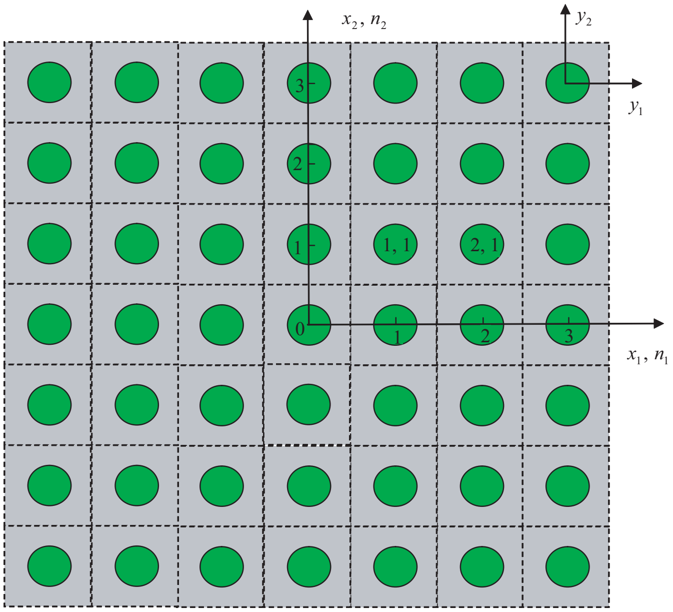

Fig. 1. Coordinate systems used in VAMUCH formulation. (only a 2D square array is drawn for clarity)

$$\Pi = \sum_{n=-\infty}^{\infty} \int_{\Omega} \frac{1}{2} C_{ijkl}(y_1, y_2, y_3)\ \epsilon_{ij}\ \epsilon_{kl} \mathrm{d}\Omega \tag{2}$$

where C_{ijkl} are the components of the microscopically varying fourth-order elasticity tensor and ϵ_{ij} are the components of the 3D strain tensor defined for linear theory as

$$\epsilon_{ij}(\mathbf{n}; \mathbf{y}) = \frac{1}{2} \left[\frac{\partial u_i(\mathbf{n}; \mathbf{y})}{\partial y_j} + \frac{\partial u_j(\mathbf{n}; \mathbf{y})}{\partial y_i} \right] \tag{3}$$

Here $u_i(\mathbf{n}; \mathbf{y})$ are functions of the integer coordinates and the local coordinates for each UC. In view of the fact that the infinitely many UCs form a continuous heterogeneous material, we need to enforce the continuity of the displacement field u_i on the interface between adjacent UCs. To avoid the difficulty associated with integer variables, we can transform this variational problem into a more convenient format using the idea of a quasicontinuum [16]. The basic idea is to associate a function of integer arguments defined in the integer space with a continuous function defined in $\mathcal{R}$.

In view of assumption 1 given in Eq. (1), we can make the following change of variables

$$u_i(\mathbf{x}; \mathbf{y}) = v_i(\mathbf{x}) + y_j \frac{\partial v_i}{\partial x_j} + \chi_i(\mathbf{x}; \mathbf{y}) \tag{4}$$

with χ_i denoted as the fluctuating functions which are believed to be asymptotically smaller than v_i because the material is statistically homogeneous, *i.e.*, $\chi_i \sim o(v_i)$. It is noted that $\langle \chi_i \rangle = 0$ if the origin of the y_i coordinate system is chosen to be the center of UC. Then we can apply VAM to obtain the first-order asymptotic expansion of the variational statement and obtain the following functional defined for a single UC.

$$\mathit{\Pi}_\Omega = \frac{1}{2\Omega} \int_\Omega C_{ijkl} \left[\bar{\epsilon}_{ij} + \frac{1}{2}(\chi_{i,j} + \chi_{j,i}) \right] \left[\bar{\epsilon}_{kl} + \frac{1}{2}(\chi_{k,l} + \chi_{l,k}) \right] \mathrm{d}\Omega \tag{5}$$

with $\bar{\epsilon}_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$ denoting the macroscopic strain field of the homogenized material. The VAMUCH theory can be stated as minimizing the functional $\mathit{\Pi}_\Omega$ subject to the following constraints

$$\chi_i(\mathbf{x}; d_1/2, y_2, y_3) = \chi_i(\mathbf{x}; -d_1/2, y_2, y_3) \tag{6}$$

$$\chi_i(\mathbf{x}; y_1, d_2/2, y_3) = \chi_i(\mathbf{x}; y_1, -d_2/2, y_3) \tag{7}$$

$$\chi_i(\mathbf{x}; y_1, y_2, d_3/2) = \chi_i(\mathbf{x}; y_1, y_2, -d_3/2) \tag{8}$$

$$\langle \chi_i \rangle = 0 \tag{9}$$

where Eqs. (6)-(8) are the well-known periodic boundary conditions of MHT and Eq. (9) helps to uniquely determine the fluctuating functions χ_i.

This constrained minimization problem can be solved analytically for very simple cases such as binary composites. For general cases we need to turn to computational techniques for numerical solutions. Since the VAMUCH theory is inherently variational, the well-established finite element method is a natural choice as a numerical technique to solve this problem. The details of finite element implementation are given in VAMUCH publications [27, 28, 21, 22, 23]. As a result, a companion computer program VAMUCH has been developed as a general-purpose micromechanical analysis code.

It is pointed out that VAMUCH provides a very general micromechanics modeling framework for heterogeneous materials. To model behavior other than linear elastic behavior, we just need to change the energy expression in Eq. (2). For example, if we want to model the thermoelastic behavior, we need to use the Helmholtz free energy [28]; if we want to model the effective thermal conductivity, we need to use an energy integral with temperature as the field variable [21]; if we want to model electromagnetoelastic behavior, we need to use the electromagnetic enthalpy in Eq. (2) with displacement, electric field and magnetic field as the the field variables [22]. Now VAMUCH can perform a multiphysics micromechanics modeling of heterogeneous material to predict the effective linear, multiphysics properties along with the corresponding local fields within the microstructure [25]. The VAMUCH modeling framework can also be applied to model nonlinear material behavior. For example, to model the elastoplastic behavior, we need to use the energy changes in terms of strain increments in Eq. (2) [24]. Although whether the VAMUCH framework can be applied to model defects in the microstructure is still under investigation, there is no doubt that VAMUCH can be applied to model both linear and nonlinear behavior of heterogeneous materials.

3 Uniqueness of VAMUCH

In addition to providing a general framework for modeling heterogeneous materials, VAMUCH has several unique features in both its theory and application in comparison to other micromechanics approaches.

3.1 Unique features in VAMUCH theory

Taking advantage of the smallness of the microstructure of heterogeneous materials, VAMUCH formulates a variational statement of the unit cell through an asymptotic analysis of the energy functional by invoking only two very basic assumptions of micromechanics. Other convenient assumptions commonly used in the literature are avoided.

It can be shown that the governing differential equations of MHT, which achieves the best available accuracy for periodic composites, can be derived from the variational statement of VAMUCH [27]. The main differences between VAMUCH and MHT are:

- The periodic boundary conditions are derived in VAMUCH, while they are assumed *a priori* in MHT. MHT also assumes periodic functions, which is shown to be unnecessary in VAMUCH.
- The fluctuating functions are determined uniquely in VAMUCH, while they can only be determined up to a constant in MHT.
- VAMUCH has an inherent variational nature which is convenient for numerical implementation, while virtual quantities should be carefully chosen to make MHT variational as shown in [10].

Although it is easy to distinguish VAMUCH from other analytical micromechanics approaches, VAMUCH is often confused as one of the FEA-based micromechanics approaches because the equations of the VAMUCH theory are solved using the finite element technique. FEA-based micromechanics approaches carry out a conventional finite element analysis of an RVE (or UC) with specially designed boundary conditions under specifically designed loads. Although VAMUCH has the same versatile modeling capability as FEA-based approaches, VAMUCH is dramatically different from FEA-based approaches, both in its theory and application.

To distinguish the theoretical differences between VAMUCH and FEA-based approaches, we need to consider the corresponding differential statements of VAMUCH and FEA-based approaches. The corresponding differential statement of VAMUCH for elastic materials includes the following governing differential equation

$$\frac{\partial}{\partial y_l} C_{ijkl} \left[\bar{\epsilon}_{ij} + \frac{1}{2}(\chi_{i,j} + \chi_{j,i}) \right] = 0 \quad \text{in} \quad \Omega \tag{10}$$

along with the following periodic stress boundary conditions

$$C_{ijkl}[\bar{\epsilon}_{ij} + \frac{1}{2}(\chi_{i,j} + \chi_{j,i})]_{y_1=d_1/2} = C_{ijkl}[\bar{\epsilon}_{ij} + \frac{1}{2}(\chi_{i,j} + \chi_{j,i})]_{y_1=-d_1/2} \tag{11}$$

$$C_{ijkl}[\bar{\epsilon}_{ij} + \frac{1}{2}(\chi_{i,j} + \chi_{j,i})]_{y_2=d_2/2} = C_{ijkl}[\bar{\epsilon}_{ij} + \frac{1}{2}(\chi_{i,j} + \chi_{j,i})]_{y_2=-d_2/2} \tag{12}$$

$$C_{ijkl}[\bar{\epsilon}_{ij} + \frac{1}{2}(\chi_{i,j} + \chi_{j,i})]_{y_3=d_3/2} = C_{ijkl}[\bar{\epsilon}_{ij} + \frac{1}{2}(\chi_{i,j} + \chi_{j,i})]_{y_3=-d_3/2} \tag{13}$$

the periodic boundary conditions for fluctuating functions in Eqs. (6)-(8), and the uniqueness condition for the fluctuating functions in Eq. (9).

The governing differential equation of FEA-based approaches for elastic properties is the 3D equilibrium equation without body force

$$\frac{\partial}{\partial y_l} C_{ijkl} \left(u_{i,j} + u_{j,i} \right) = 0 \quad \text{in} \quad \Omega \tag{14}$$

Comparing this equation with the governing differential equation of VAMUCH in Eq. (10), one clearly observes that the fundamental variables of VAMUCH are fluctuating functions while those of FEA-based approaches are the macroscopic displacements. Furthermore, the boundary conditions for FEA-based approaches are applied on the macroscopic variables such as displacements. Different sets of displacement boundary conditions are needed for calculating different properties. Since these boundary conditions are applied a priori based on engineering intuition, it is not surprising that different researchers introduced different boundary conditions for calculating the same property, see [20] for a detailed discussion on the boundary conditions for RVEs. It is known that the predicted effective properties are very sensitive to boundary conditions. Another theoretical difference is that the dimensionality of VAMUCH analysis is based on the periodicity of the microstructure. For example, we can use 1D UC to model binary composites, 2D UC to model fiber reinforced composites, and 3D UC to model particle reinforced composites. No special treatment is necessary for these different types of microstructures. However, it is not the case with FEA-based approaches, one has to use 3D UCs to get the complete set of 3D material properties, whether it be a binary composite, fiber reinforced composite, or particle reinforced composite. For example, according to the authors' understanding, Sun and Vaidya [20] derived the most rigorous FEA-based approach for elastic properties, which requires a 3D RVE for fiber reinforced composites.

4 Unique Features in VAMUCH Application

Although there are significant theoretical differences between VAMUCH and other micromechanics approaches, practicing engineers are often more concerned with convenience and efficiency. To this end, we compare VAMUCH and FEA-based approaches. To use a FEA-based approach, one has to carry out multiple runs with different sets of boundary conditions and external loads for predicting different material properties. And postprocessing steps such as averaging stresses or averaging strains are needed for calculating the effective properties. If one is also interested in the local fields within the microstructure, one more run is necessary to predict local stress/strain field if the global stress/strain state is different from that used to obtain the effective properties. Comparing to FEA-based approaches, VAMUCH has the following unique features:

1. VAMUCH can obtain the complete set of material properties within one analysis without applying any load and any boundary conditions, which is far more efficient and less labor intensive than those approaches requiring multiple runs under different boundary and load conditions. It is also noted that VAMUCH can even obtain the complete set of 3D material properties using a one-dimensional analysis of the 1D UC for binary composites. It is impossible for FEA-based approaches.
2. VAMUCH calculates effective properties and local fields directly with the same accuracy as the fluctuating functions. No postprocessing calculations which introduce more approximations, such as averaging stress or strain fields, are needed, which are indispensable for FEM-based approaches.
3. VAMUCH can recover the local fields using a set of algebraic relations obtained in the process of calculating the effective properties. Another analysis of the microstructures which is needed for FEA-based approaches is not necessary for VAMUCH.

Here it is worthwhile to point out that most of these features in VAMUCH application are actually not unique to VAMUCH and are shared by the method of cells (MOC) and its variants developed by Prof. Aboudi and his colleagues. The main difference between VAMUCH and MOC, as far as application is concerned, is that VAMUCH takes full advantage of the finite element technique including versatile discretization capability for arbitrary microstructure, highly efficient linear solvers, and well-developed preprocessing and postprocessing capabilities. An extensive assessment of VAMUCH, MOC and its variants, and FEA-based micromechanics approaches can be found in [29].

It is also emphasized here that VAMUCH calculations are conceptually different from automating the multiple runs including postprocessing steps of FEA-based approaches using a macro language such as APDL of ANSYS. VAMUCH is not just a different postprocessing approach.

5 A Practical Example Using VAMUCH

Micromechanics analysis is usually carried along with a macroscopic structural analysis. In this section, we will use an example to demonstrate how VAMUCH would be utilized in analyzing real engineering structures instead of a single

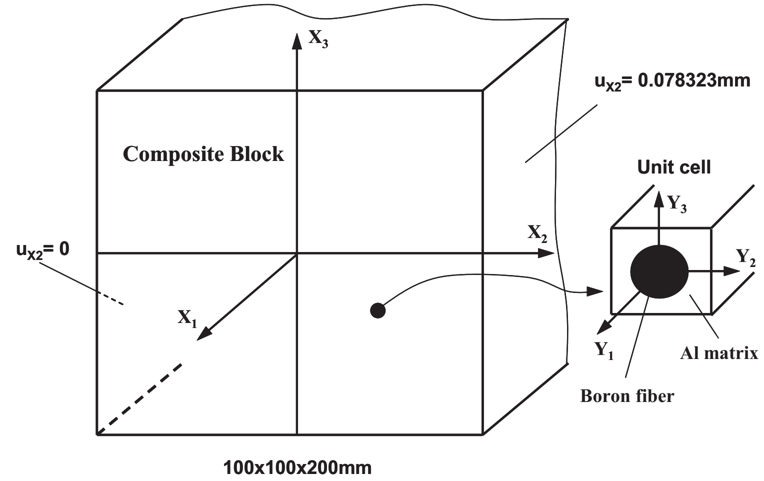

Fig. 2. Composite block subjected to uniform tensile displacement at the free boundary surface in the X_2-direction

unit cell. The example is a composite block made of boron fibers reinforced aluminum matrix as shown in Fig. 2, where we use (X_1, X_2, X_3) as the global coordinates to describe the macroscopic structure and (Y_1, Y_2, Y_3) parallel to (X_1, X_2, X_3) as the local coordinates to describe the unit cell. The block is 100 mm along the X_2 and X_3-directions and 200 mm along the X_1-direction. The number of unit cells (thus the size of the microstructure) comprising the composite block will be varied to study the limitation of the micromechanics approach. The unit cell is formed by two constituents which are isotropic with Young's modulus $E_f = 379.3$ GPa and Poisson's ratio $\nu_f = 0.1$ for the boron fibers, and Young's modulus $E_m = 68.3$ GPa and Poisson's ratio $\nu_m = 0.3$ for the aluminum matrix. The boron fiber is arranged in a square array and the volume fraction of the fibers is 0.4. The axial direction of the fiber is along the X_1-direction. A uniform displacement of 0.078323 mm is applied at the right free boundary surface of the block. The X_2 displacement component of the left boundary surface of the block is set to zero, namely, $u_{X_2} = 0$. We are interested in solving this problem using VAMUCH combined with a structural analysis.

First, we need to use VAMUCH to calculate the effective material properties based on a finite element mesh of a typical unit cell as shown in Fig. 3. Note that for this example, only a two-dimensional unit cell is needed because the periodicity of the composite is two-dimensional and the microstructure remains the same along the X_1-direction. The composite is tetragonal with the effective properties predicted by VAMUCH as $E_{11} = 193.53$ GPa, $E_{22} = E_{33} = 127.676$ GPa, $G_{12} = G_{13} = 48.304$ GPa, $G_{23} = 41.703$ GPa, $\nu_{12} = \nu_{13} = 0.209$, and $\nu_{23} = 0.278$, where ν_{ij} is defined as the negative of the normal strain along the X_i-direction divided by the normal strain along the

X_j-direction when the material is loaded along the X_j-direction. Then we use ANSYS to analyze the composite block with the same boundary conditions but the heterogeneous material is replaced with a homogenous material with the predicted effective properties. Fig. 4 shows the contour plot of the X_2-component of the displacement of the composite block made of the effective homogeneous material. Finally, if we also want to know the pointwise information of the mechanical field of this composite block, we need to recover the local fields based on the global response we just obtained. For instance, Fig. 5 shows the detailed distribution of von Mises stress at an arbitrary macro material point, which corresponds to a unit cell at the micro level. This completes all the steps necessary for one to take advantage of micromechanical approaches, of which VAMUCH is one particular method, to analyze structures made of heterogeneous materials.

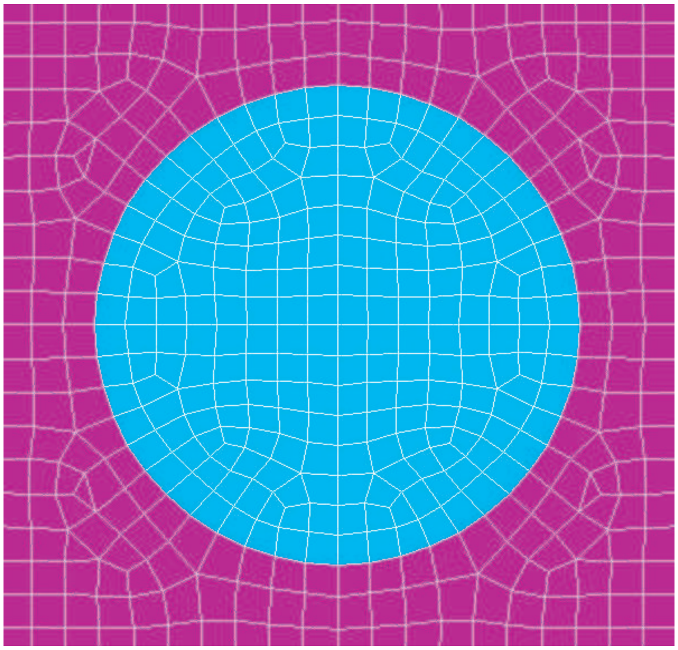

Fig. 3. Finite element mesh of the unit cell of a B/Al composite

The micromechanical approach enables the analyst to circumvent the time-consuming modeling and computations based on a finite element model including all the details of all the microstructures at the macro level structural analysis. To assess its efficiency and accuracy, we also carried out a direct analysis of the composite block made of heterogeneous material using ANSYS. As shown in Table 1, with an increasing number of unit cells, the average values of the strain energy and σ_{22} predicted by the direct analysis converge to those

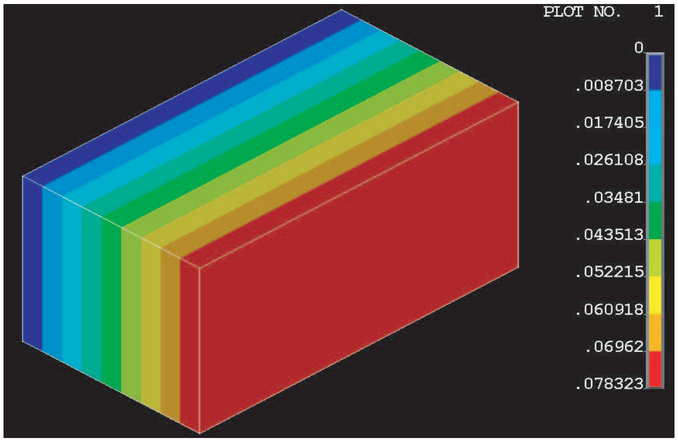

Fig. 4. Contour plot of X_2 displacement component (mm) of the effective homogeneous materials

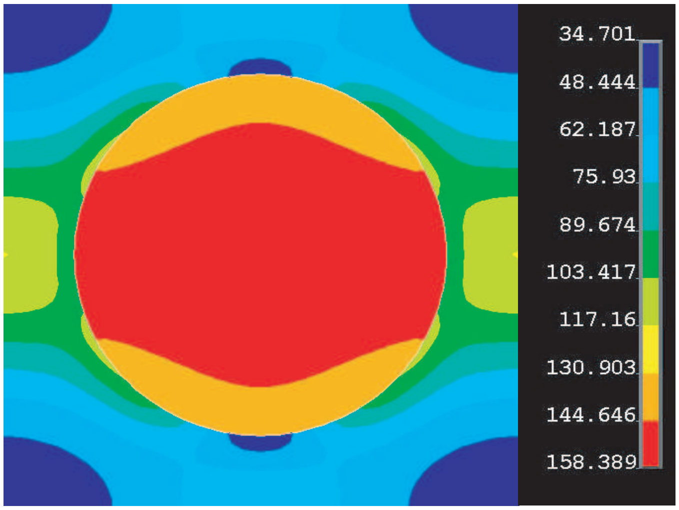

Fig. 5. Contour plot of von Mises stress (MPa) at an arbitrary macro material point

predicted by the analysis of an effective homogeneous material with the elastic constants obtained from VAMUCH used in a full finite element analysis. This implies that the accuracy of VAMUCH increases as the ratio of the size of the unit cell to that of the macroscopic structure decreases. Note as the overall size of the composite block remains the same, increasing the number of unit cells is equivalent to decreasing the size of unit cells. Although the computing time for the VAMUCH-based structural analysis of effective homogeneous material remains the same, the direct analysis takes longer when the body is composed of more unit cells.

Table 1. Comparison of average value of strain energy, σ_{22} and computing time for the composite block having different number of UCs

Number of UCs	1UC	36UCs	81UCs	Equiv. Hom.
strain energy ($\times 10^4$ J)	7.483	7.769	7.8	7.83
averaged value of σ_{22} (MPa)	95.54	99.2	99.44	100
computing time (seconds)	20	140	320	2

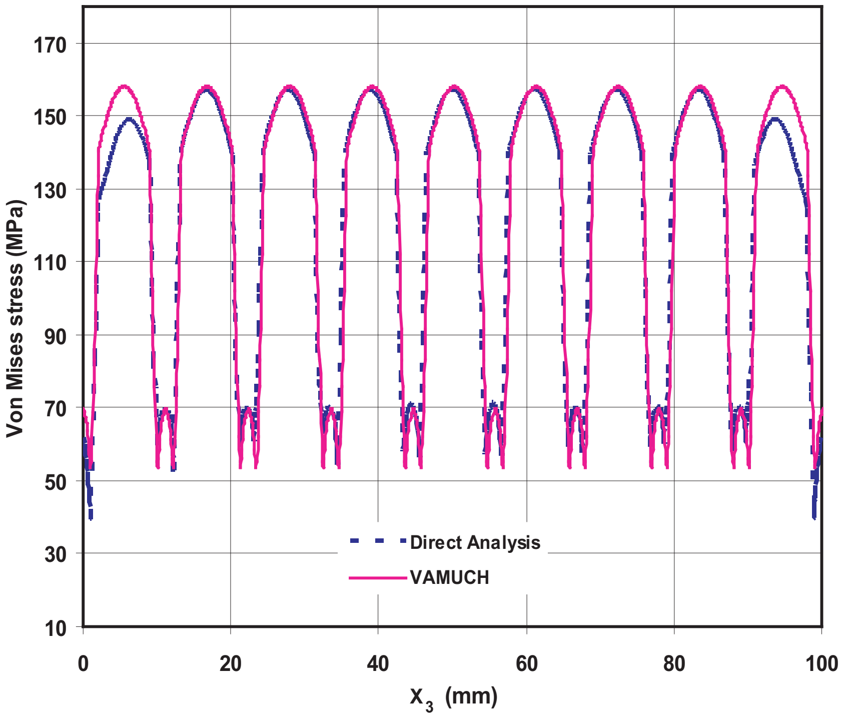

Fig. 6. The distribution of von Mises stress along the X$_3$ axis

To demonstrate whether the pointwise distribution of local fields recovered by VAMUCH is predictive, we also plot the distribution of the von Mises stress along the X_3-axis (with $X_2 = 0$) in Fig. 6 for the composite block with 81 unit cells, where the dashed line represents the prediction from the direct analysis of the heterogeneous block and the solid line represents the prediction from VAMUCH-based structural analysis. The agreement of the two predictions is very satisfying except at the boundary where the field does not exhibit a representative behavior and the VAMUCH-based structural analysis fails to capture this anomaly. However, when considering the 160 times saving of computing time (see Table 1), the VAMUCH-based structural analysis is attractive to engineers despite the slight loss of accuracy. Particularly, real engineering structures made of heterogeneous materials usually contain thousands to millions of unit cells and the microstructures might be more complex, that is, the microstructure might not be a simple square array with just two constituents. Direct analysis of such structures including all the microstructural details in the macroscopic structural analysis becomes virtually impossible for engineers who have access to only moderate computing resources and the VAMUCH-based structural analysis will become the method of choice because VAMUCH not only accurately homogenizes the heterogenous material but also accurately recovers the local fields within the microstructure.

6 Conclusion

Although VAMUCH is still in its early stage of development, it is emerging as a general-purpose micromechanics analysis tool. It can achieve the same accuracy as mathematical homogenization theory because it invokes only two very basic assumptions within the micromechanics concept. As far as modeling capability is concerned, VAMUCH is as versatile as FEA-based approaches because it can deal with arbitrary UCs containing an arbitrary number of inclusions with arbitrary shapes made of general anisotropic material, although VAMUCH is much more convenient and efficient than FEA-based approaches. In fact, one just needs to provide a mesh with corresponding constituent properties, VAMUCH will produce the complete set of material properties with one run, which takes just a very small fraction of both the model preparation time and the computational time of a FEA-based approach. Also to obtain the complete set of properties of fiber reinforced composites or binary composites, FEA-based approaches need to use a 3D UC, while VAMUCH will only need to use a 2D UC and a 1D UC, respectively. The time saving by this dimensionality reduction is dramatic.

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References

1. Aboudi J (1982) Int J Eng Sci 20: 605–621
2. Aboudi J (1989) Appl Mech Rev 42:193–221
3. Aboudi J, Pindera MJ, Arnold SM (2001) J Appl Mech 68:697–707
4. Accorsi ML, Nemat-Nasser, S (1986) Mech Mater 5:209 – 220
5. Banerjee B, Adams DO (2004) Int J Solids Struct 41:481–509
6. Bensoussan A, Lions J, Papanicolaou G (1978). Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam
7. Berdichevsky VL (1977) PMM 41: 993–1006
8. Berdichevsky VL (1979) PMM 43: 664–687
9. Dvorak GJ, Bahei-El-Din, YA (1979) J Mech Phys Solids 27:51–72
10. Guedes JM, Kikuchi N (1990) Comput Method Appl M 83: 143–198
11. Hashin Z (1983) J Appl Mech 50:481–505
12. Hashin Z, Shtrikman S (1962) J Mech Phys Solids 10:343–352
13. Hill R (1952) Proc. Phys. Soc. London A65:349–354
14. Hill R (1963) J Mech Phys Solids 11:357–372
15. Hill R (1965) J Mech Phys Solids 13:189–198
16. Kunin I (1982) Theory of Elastic Media with Microstructure. Vol. 1 and 2. Springer Verlag
17. Milton GW (1981) Phys Rev Lett 46:542–545
18. Nemat-Nasser S, Hori M (1993) Micromechanics: Overall Properties of Heterogeneous Materials, 2nd ed., North-Holland, Amsterdam
19. Paley M, Aboudi J (1992) Mech Mater 14:127–139
20. Sun CT, Vaidya RS (1996) Compos Sci Technol 56:171–179
21. Tang T, Yu W (2007) J Mech Mater Struct 2: 1813–1830
22. Tang T, Yu W (2008) Int J Eng Sci 46: 741–757
23. Tang T, Yu W (2008) Mech Mater 40: 812–824
24. Tang T, Yu W (2008) Mech Adv Mater Struct , submitted
25. Tang T, Yu W (2009) Smart Mater Struct, submitted
26. Williams TO (2005) Int J Solids Struct 42:1009–1038
27. Yu W, Tang T (2007) Int J Solids Struct 44: 3738–3755
28. Yu W, Tang T (2007) Int J Solids Struct 44: 7510-7525
29. Yu W, Williams TO, Bednarcyk BA, Aboudi J, Tang T (2007), Proceedings of the 48th Structures, Structural Dynamics and Materials Conference, April 23–26, Waikiki, Hawaii